Comprehensive Final Exam
SUGGESTED ANSWERS

True/False/Ambiguous: provide an intuitive explanation that the following statement is true, false or uncertain. Simply stating the belief will yield little credit (7.5 points each)

1. That a function is continuous is necessary and sufficient for it to be differentiable.

False. A function must be continuous for it to be differentiable, but a function may be continuous and not differentiable (e.g. absolute value function), thus continuity is not a sufficient condition for differentiability.

2. Consumer surplus is the difference in the area between the supply curve and the equilibrium price and represents the amount consumers spend more than they have to.

False. CS is the area between the demand curve and the equilibrium price and it represents the gain consumers earn because they do not spend as much for a good as they were willing to.

Very Short Problems (7.5 points each)
3. a. Express the rate of change of k (capital per worker) starting with the definition of k = K/L. Show all your steps.

b. Use the rate of change of aggregate capital ($\dot{K} = sLk^{\propto}$) and $\frac{\dot{L}}{L} = n$ to insert into the result from part a to express the evolution of per-capita capital as in the Solow growth model.

Answer:

Take logs of both sides $\ln k = \ln K - \ln L$

Take time derivative $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$

Substitute $\frac{\dot{k}}{k} = \frac{sLk^{\propto}}{K} - n$

Simplifying $\dot{k} = sk^{\propto} - nk$

4. The average-cost AC function is $AC = Q^2 - 4Q + 214$

Find an expression for marginal cost from the related total cost function, then find output Q where AC reaches a minimum. What is this minimum value of AC?

Answer:

$$TC = AC \cdot Q = (Q^2 - 4Q + 214)Q = Q^3 - 4Q^2 + 214Q$$

$$MC = \frac{dTC}{dQ} = 3Q^2 - 8Q + 214$$

$$AC_{\min} = MC$$

$$Q^2 - 4Q + 214 = 3Q^2 - 8Q + 214$$

$$2Q^2 - 4Q = 0$$

$$Q_1 = 0 \qquad Q_2 = 2$$

You can compare the AC function with the two critical values to find that AC is minimized at $Q_2 = 2$

Short Problems (10 points each)

5. Taylor expansion: You need to find $f(x) = \exp(0.5 * x)$ for x = 1.1. Use the Taylor expansion to approximate this function around a = 1. Round to three digits in the following approximations. Find the correct value for the function and compare with the approximations.

Calculate the approximations for f(x) around a = 1 using
a) the constant function;
b) 1st order Taylor series
c) 2nd order Taylor series.

a) Constant value function: $f(1) = \exp(0.5 * x) = 1.649$

b) 1st order approximation: $g(x) = f(a) + f'(a) \cdot (x - a)$

$f'(x) = 0.5e^{0.5x}$ => $f'(1) = 0.824$

$g(x) = 1.649 + 0.824 \cdot (0.1) = 1.731$

c) 2nd order approximation: $g(x) = f(a) + f'(a) \cdot (x - a) + \frac{f''(a)}{2}(x - a)^2$

$f''(x) = 0.25e^{0.5x}$ => $f''(1) = 0.412$

$g(x) = 1.731 + 0.412 \cdot (0.01)/2 = 1.733$

The exact value of $f(x) = \exp(0.55) = 1.733$ *so it seems a fair approximation.*

6. Assuming all parameters are positive, let demand and supply in a market be:

$$Q^D = \alpha - \beta p - \eta \frac{dp}{dt}$$

$$Q^S = \delta p$$

Assuming that the market is in equilibrium at every point in time, find the time path p(t) (the general solution). Does this market have an dynamically stable intertemporal equilibrium price?

ANSWERS:

Set the two equations equal: $\delta p = \alpha - \beta p - \eta \frac{dp}{dt}$

Rearrange: $\frac{dp}{dt} + p\left(\frac{\beta + \delta}{\eta}\right) = \frac{\alpha}{\eta}$

Particular integral gives $p = \frac{\alpha}{\beta + \delta}$ *(this is the intertemporal equilibrium)*

Complementary function: $\frac{dp}{dt} + p\left(\frac{\beta+\delta}{\eta}\right) = 0$ => $p_c = Ae^{-\left(\frac{\beta+\delta}{\eta}\right)t}$

Since the complementary function will disappear due to the negative exponent term, the general solution is dynamically stable.

7. A firm producing two goods with quantities Q_i where i = 1, 2 has a profit function $P_1Q_1 + P_2Q_2 - T(Q_1, Q_2)$, where P_i are the given prices and T is the total cost function. Assume $T_1, T_2 > 0$ and also that $T_{11}, T_{22} > 0$.

a. Write the first order conditions for a maximum. Give these conditions an economic interpretation.

b. Place the second order conditions in a matrix and show that this is a maximum. What additional assumption must you make to be definite?

Answers:

a. *The FOC:* $P_1 - T_1 = 0$ *and* $P_2 - T_2 = 0$ *In words the condition is that the marginal revenue must equal marginal cost for each good.*

b. *The Hessian matrix for unconstrained maximum is* $\begin{vmatrix} -T_{11} & -T_{12} \\ -T_{21} & -T_{22} \end{vmatrix}$. *The first element is negative and the determinant is* $T_{11}T_{22} - T_{12}^2$. *To have a negative definite matrix the equare of the cross partials must be smaller than the product of the good 1 & 2 2nd partials (i.e. the determinant must be positive)*

Problems: (14 points each)

8. Markov Chains. At time t, 3 rich persons exist and 97 poor people also exist. The probability of a rich person today becoming poor next period is .03. The probability of a person who is poor today becoming rich tomorrow is .02.

a. Set up a transition matrix and find the number of rich and poor people at t + 1. Give your answer in tenths of persons.

b. Solve for the long-run equilibrium probabilities using an uncoupled transformation of the matrix.

Answers:

a. $x_{t+1} = \begin{bmatrix} 0.97 & 0.02 \\ 0.03 & 0.98 \end{bmatrix}\begin{bmatrix} R \\ P \end{bmatrix}$ *or* $x_{t+1} = \begin{bmatrix} .97 & .02 \\ .03 & .98 \end{bmatrix}\begin{bmatrix} 3 \\ 97 \end{bmatrix} = \begin{bmatrix} 4.85 \\ 95.15 \end{bmatrix}$

b. *We know one eigenvalue is 1 and the other is the trace – 1. Thus the other eigenvalue is 0.97 + 0.98 – 1 = 0.95. The eigenvectors for λ = 1 is*

$x_{t+1} = \begin{bmatrix} -0.03 & 0.02 \\ 0.03 & -0.02 \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = 0$ *A solution is* $p_{11} = 2$ *and* $p_{21} = 3$.

a. *The eigenvectors for λ = 0.95 is*

$$x_{t+1} = \begin{bmatrix} 0.02 & 0.02 \\ 0.03 & 0.03 \end{bmatrix} \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} = 0$$ *A solution is p_{12} = 1 and p_{22} = -1.*

Thus the general solution is $\begin{bmatrix} R_t \\ P_t \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} (0.95)^t$

The system is stable since the second terms go to zero as t gets very large. We know that c1 = 1/(2+3) = 1/5 so the long term probabilities are 0.40 of being rich and 0.60 of being poor.

```
>> A = [.97 .02; .03 .98];
>> A10 = A^10
A10 =
   0.7592   0.1605
   0.2408   0.8395
>> A20=A^20
A20 =
   0.6151   0.2566
   0.3849   0.7434
>> A30 = A^30
A30 =
   0.5288   0.3141
   0.4712   0.6859
>> A200 = A^200
A200 =
   0.4000   0.4000
   0.6000   0.6000
```

9. A simple model describes equilibrium in the market for one good:

$$Q^D = D(P, Y_0) \qquad D_P < 0;\ D_Y > 0$$

$$Q^S = S(P, A_0) \qquad S_P > 0;\ S_A > 0$$

Where Q^D, Q^S are quantity demanded and supplied. Y is income and A is technology.

(a) Write the equilibrium conditions in a system of equations and place in matrices.

(b) Find $\partial Q^* / \partial A_0$ *and* $\partial P^* / \partial A_0$. Assign signs to these comparative static derivatives. For an improvement in technology find how Q and P react in equilibrium.

(a) The system is, after dividing by exogenous variable dA_0 $\begin{bmatrix} 1 & -D_P \\ 1 & -S_P \end{bmatrix} \cdot \begin{bmatrix} \frac{dQ^*}{dA_0} \\ \frac{dQ^*}{dA_0} \end{bmatrix} = \begin{bmatrix} 0 \\ S_A \end{bmatrix}$

(b) ***Jacobian:*** $|J| = \begin{vmatrix} 1 & -D_P \\ 1 & -S_P \end{vmatrix} = -S_P + D_P < 0$ *(so we can apply the IFT)*

Use Cramer's Rule: $\dfrac{dQ^*}{dA_0} = \dfrac{\begin{vmatrix} 0 & -D_P \\ S_A & -S_P \end{vmatrix}}{D_P - S_P} = \dfrac{D_P S_A}{D_P - S_P} > 0$

$$\frac{dQ^*}{dA_0} = \frac{\begin{vmatrix} 1 & 0 \\ 1 & S_A \end{vmatrix}}{D_P - S_P} = \frac{S_A}{D_P - S_P} < 0$$

So this simple model predicts that improved technology (think of lower costs) results in higher quantity but lower prices in equilibrium.

10. Erich collects comic books and he favors foreign editions (F) over the domestic (D). His utility function is U = 2 ln F + ln D. He seeks to maximize utility for the $3000 income he's devoted to his collection. He must pay an average $300 for each foreign and $500 for each domestic issue. Erich's vacation for his collections is limited to 200 hours. To search for each foreign issue takes 40 hours while each domestic is easier at 20 hours.

a. Write the Lagrangian function Z including the money and time constraints.

b. Write the Z function partials for F, D and for each multiplier.

c. If the money constraint binds, find the solution for F and D. Does this solution give consistent results ?

d. If the time constraint binds, find the solution for F and D. Does this solution give consistent results ?

e. If both constraints bind, find the solution for F and D. Does this solution lead to consistent results ?

f. Find the consistent maximizing values of F and D from part c, d or e to find the maximized utility value.

Answers:

a. $Z = \ln 2F + \ln D + \lambda_1[3000 - 300F - 500D] + \lambda_2[200 - 40F - 20D]$

b. $Z_F = \frac{2}{F} - 300\lambda_1 - 40\lambda_2$

$Z_D = \frac{1}{D} - 500\lambda_1 - 20\lambda_2$

$Z_1 = [3000 - 300F - 500D]$

$Z_2 = [200 - 40F - 20D]$

c. *Money constraint binds but time does not:* $\lambda_2 = 0$ *and this implies from* $Z_F: 2/F = 300\lambda_1$ *and from* $Z_D: 1/D = 500\lambda_1$ *. This implies 150F = 500D. From* $Z_1 = 0$*, find 3000 = 300F + 150F => F = 6.67 and D = 2. But plugging these values into the time constraint gives 200 < 40(6.67) + 2(20) so it's NOT consistent.*

d. *Time constraint binds, but money does not:* $\lambda_1 = 0$ => $Z_F: 2/F = 40\lambda_2$ *and* $Z_D: 1/D = 20\lambda_2$ *Thus F = D. From* $Z_2 = 0$*, find 200 = 40F + 20D or D = 3.33 = F. Checking the money constraint, 3000 > 300 (3.33) + 500 (3.33) so it IS consistent.*

e. *Both constraints bind: set* $Z_1 = Z_2 = 0$ *& solve simultaneous equations. Find F = 2.86 and D = 4.28.*

f. *Part d utility = 2ln(3.33) + ln(3.33) =2.406 + 1.203 = 3.609. Part e utility = 2 ln (2.86) + ln (4.28) = 2.102 + 1.454 = 3.556. Thus Erich should choose the same number of domestic and foreign comics: 3.33 .*